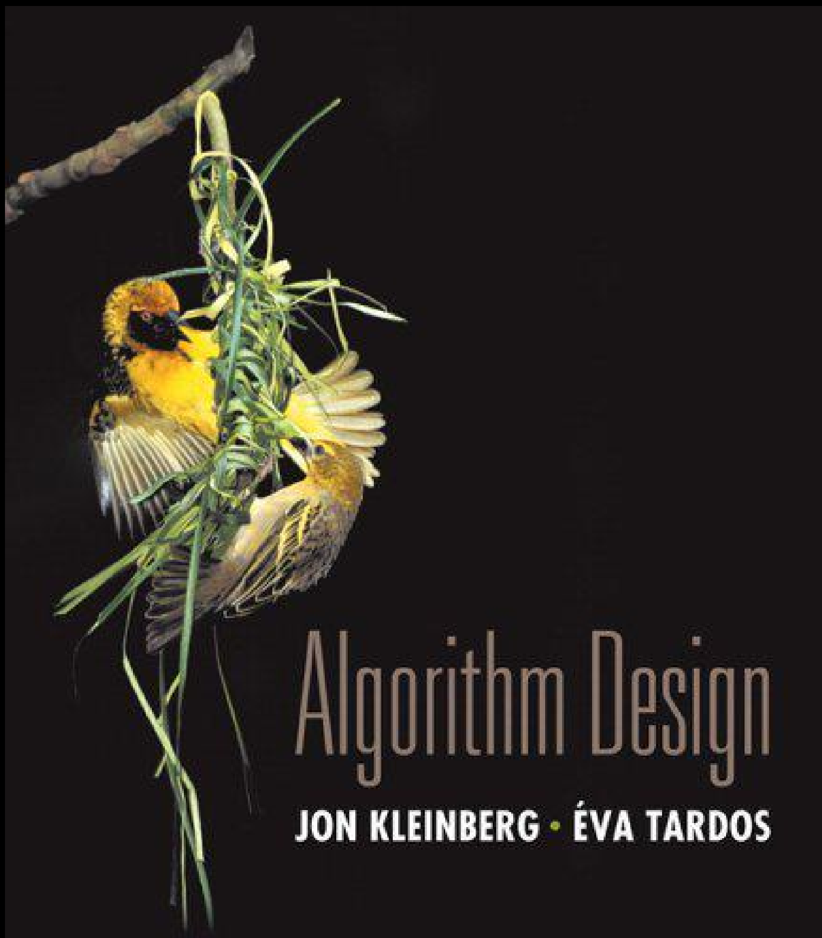


Chapter 8

NP and Computational Intractability

Slides by Kevin Wayne.

Algorithm Design Patterns and Anti-Patterns

Algorithm design patterns. Ex.

- Greed. $O(n \log n)$ interval scheduling.
- Divide-and-conquer. $O(n \log n)$ FFT.
- Dynamic programming. $O(n^2)$ edit distance.
- Reductions.

Algorithm design anti-patterns.

- NP-completeness. $O(n^k)$ algorithm unlikely.
- PSPACE-completeness. $O(n^k)$ certification algorithm unlikely.
- Undecidability. No algorithm possible.

8.1 Polynomial-Time Reductions

Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

A working definition. [Cobham 1964, Edmonds 1965, Rabin 1966]
Those with polynomial-time algorithms.

Yes	Probably no
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover

Primality testing	Factoring

Classify Problems

Desiderata. Classify problems according to those that can be solved in polynomial-time and those that cannot.

Huge number of fundamental problems have defied classification for decades.

This chapter. Show that these fundamental problems are "computationally equivalent" .

Polynomial-Time Reduction

Desiderata'. Suppose we could solve X in polynomial-time. What else could we solve in polynomial time?

Reduction. Problem X polynomial reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Notation. $X \leq_P Y$.

Remarks.

- We pay for time to write down instances sent to black box $\Rightarrow$ instances of Y must be of polynomial size.
- Note: Cook reducibility.

Polynomial-Time Reduction

Purpose. Classify problems according to relative difficulty.

Design algorithms. If $X \le_P Y$ and Y can be solved in polynomial-time, then X can also be solved in polynomial time.

Establish intractability. If $X \le_P Y$ and X cannot be solved in polynomial-time, then Y cannot be solved in polynomial time.

Establish equivalence. If $X \le_P Y$ and $Y \le_P X$, we use notation $X \equiv_P Y$.

Reduction By Simple Equivalence

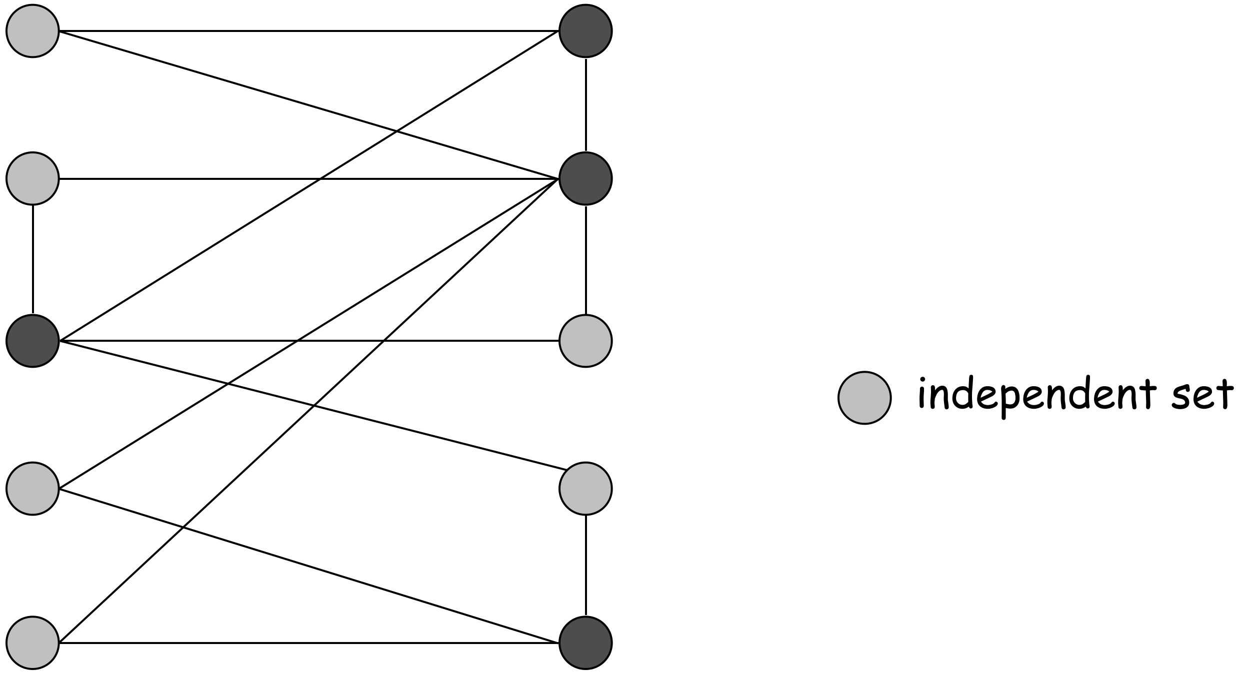
Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Independent Set

INDEPENDENT SET: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

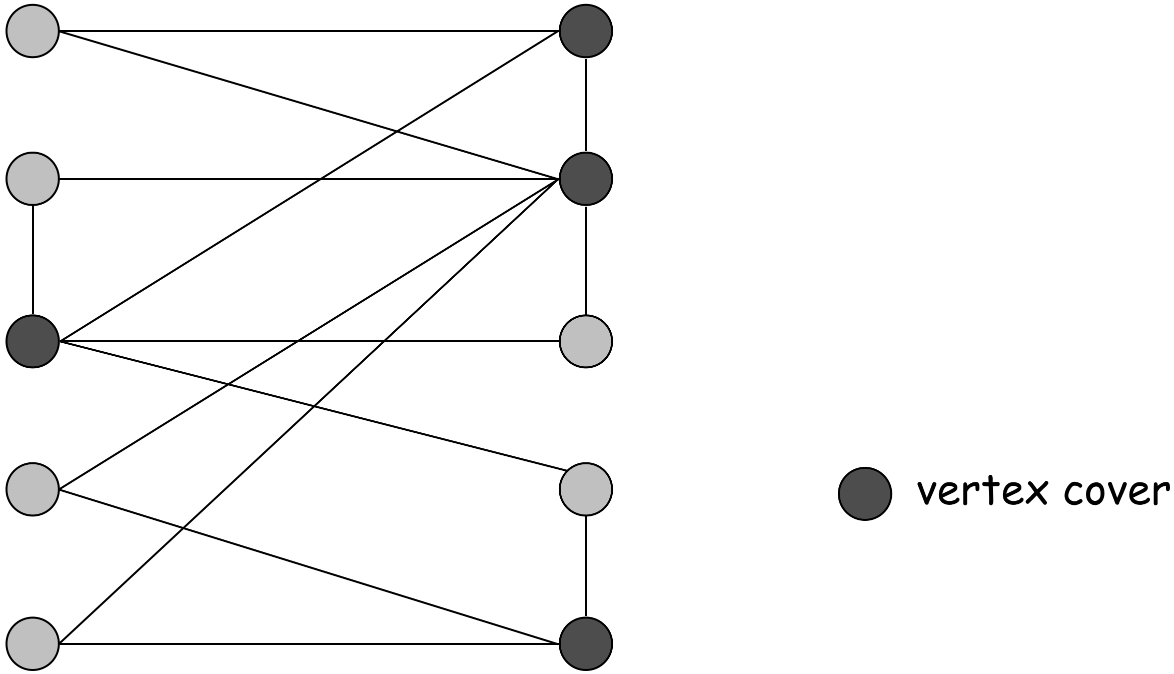
Ex. Is there an independent set of size ≥ 6? Yes.
Ex. Is there an independent set of size ≥ 7? No.



Vertex Cover

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

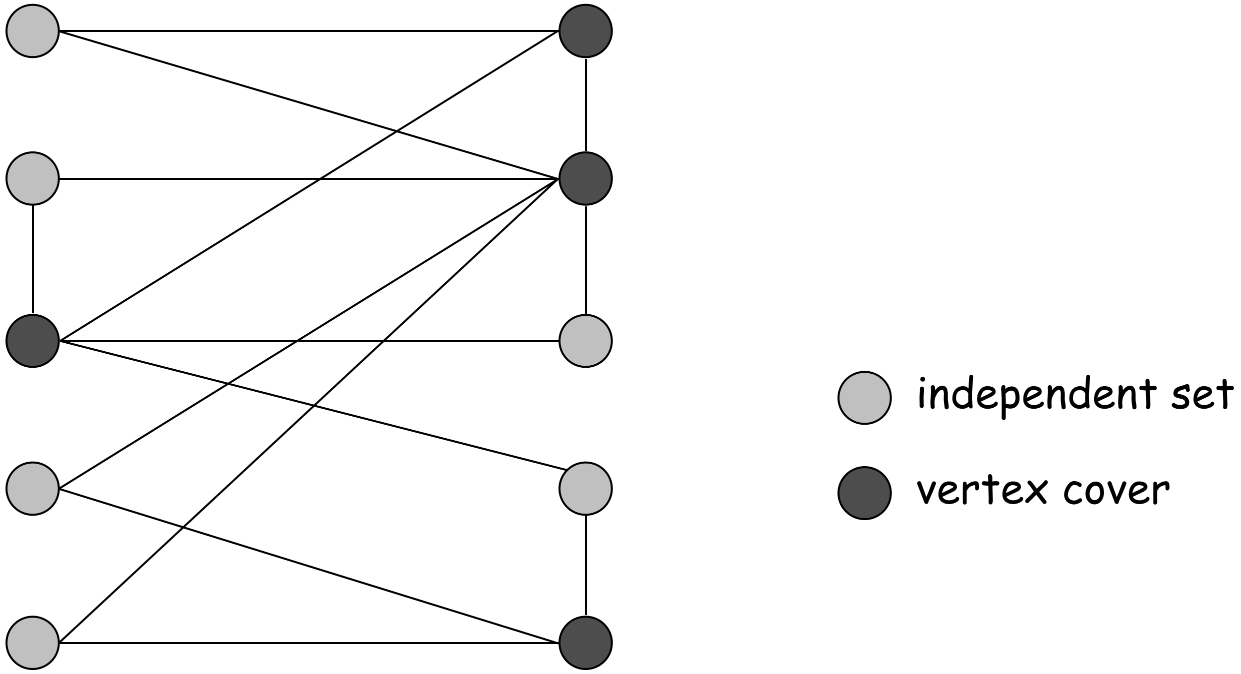
Ex. Is there a vertex cover of size ≤ 4? Yes.
Ex. Is there a vertex cover of size ≤ 3? No.



Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V – S is a vertex cover.



Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V – S is a vertex cover.

$\Rightarrow$

- Let S be any independent set.
- Consider an arbitrary edge (u, v).
- S independent $\Rightarrow$ u $\notin$ S or v $\notin$ S $\Rightarrow$ u $\in$ V – S or v $\in$ V – S.
- Thus, V – S covers (u, v).

$\Leftarrow$

- Let V – S be any vertex cover.
- Consider two nodes u $\in$ S and v $\in$ S.
- Observe that (u, v) $\notin$ E since V – S is a vertex cover.
- Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set. ▪

Reduction from Special Case to General Case

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of $\leq$ k of these sets whose union is equal to U?

Ex:

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$k = 2$

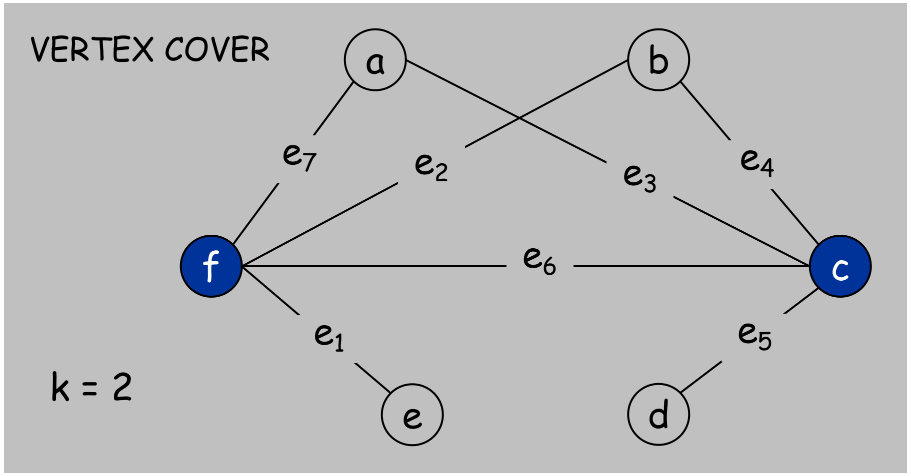
$S_1 = \{3, 7\}$	$S_4 = \{2, 4\}$
$S_2 = \{3, 4, 5, 6\}$	$S_5 = \{5\}$
$S_3 = \{1\}$	$S_6 = \{1, 2, 6, 7\}$

Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\le_P$ SET-COVER.
Pf. Given a VERTEX-COVER instance $G = (V, E)$, k, we construct a set cover instance whose size equals the size of the vertex cover instance.

Construction.

- Create SET-COVER instance:
  - $k = k$, $U = E$, $S_v = \{e \in E : e \text{ incident to } v\}$
- Set-cover of size $\le k$ iff vertex cover of size $\le k$. ▪



SET COVER

$U = \{1, 2, 3, 4, 5, 6, 7\}$
$k = 2$
$S_a = \{3, 7\}$ $S_b = \{2, 4\}$
$S_c = \{3, 4, 5, 6\}$ $S_d = \{5\}$
$S_e = \{1\}$ $S_f = \{1, 2, 6, 7\}$

8.2 Reductions via "Gadgets"

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."

Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

Ex: $\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_2 \vee x_3\right) \wedge \left(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}\right)$

Yes: x_1 = true, x_2 = true x_3 = false.

3 Satisfiability Reduces to Independent Set

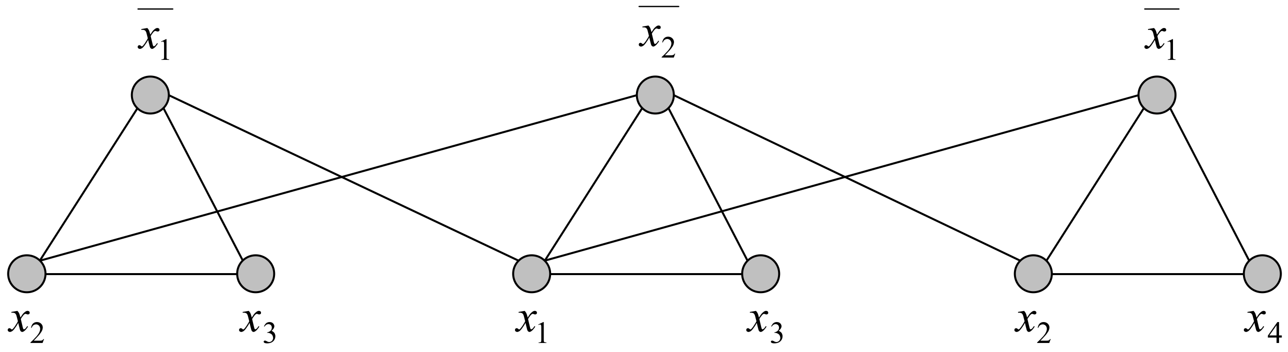
Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3 Satisfiability Reduces to Independent Set

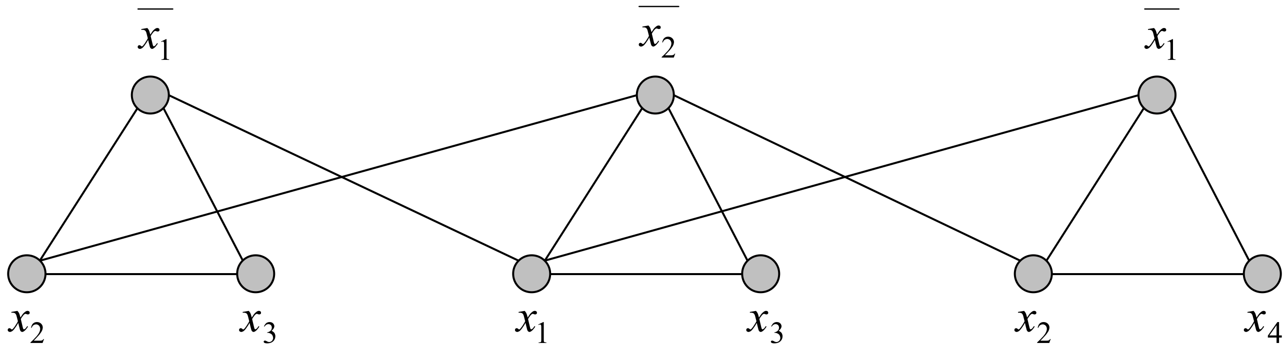
Claim. G contains independent set of size k = $|\Phi|$ iff Φ is satisfiable.

Pf. $\Rightarrow$ Let S be independent set of size k.

- S must contain exactly one vertex in each triangle.
- Set these literals to true.
- Truth assignment is consistent and all clauses are satisfied.

Pf $\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

Review

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.
Pf idea. Compose the two algorithms.

Ex: 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

Self-Reducibility

Decision problem. Does there exist a vertex cover of size $\leq$ k?

Search problem. Find vertex cover of minimum cardinality.

Self-reducibility. Search problem $\leq_P$ decision version.

- Applies to all (NP-complete) problems in this chapter.
- Justifies our focus on decision problems.

Ex: to find min cardinality vertex cover.

- (Binary) search for cardinality k* of min vertex cover.
- Find a vertex v such that $G - \{ v \}$ has a vertex cover of size $\leq$ k* - 1.
  - any vertex in any min vertex cover will have this property
- Include v in the vertex cover.
- Recursively find a min vertex cover in $G - \{ v \}$.