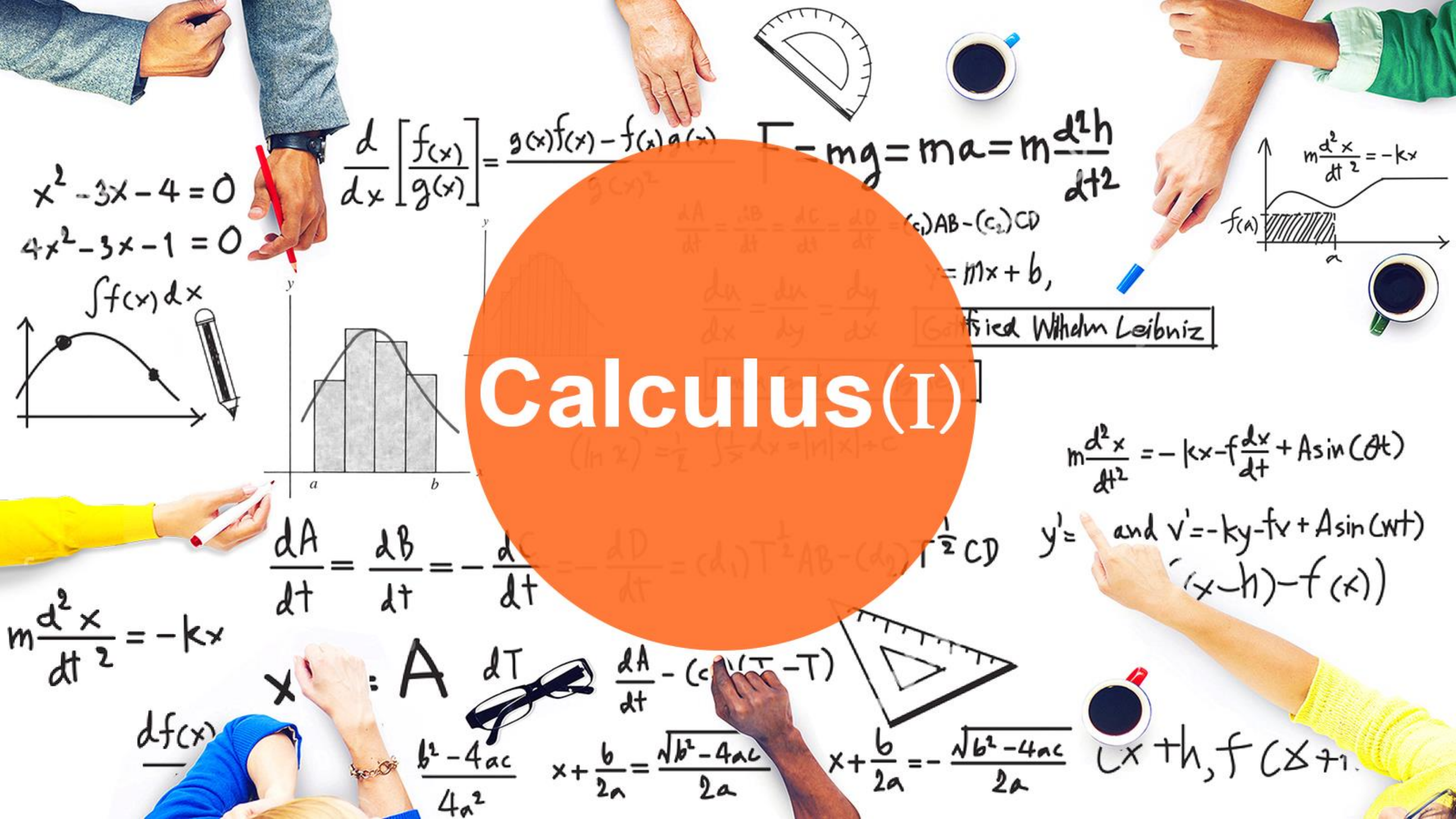
Calculus(Ⅰ)



3.8 Antiderivatives

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Definition of antiderivative and indefinite integral

Geometric problems

Let the tangent slope at any point on the curve equation be equal to twice the abscissa at the tangent point, find the equation for the curve.

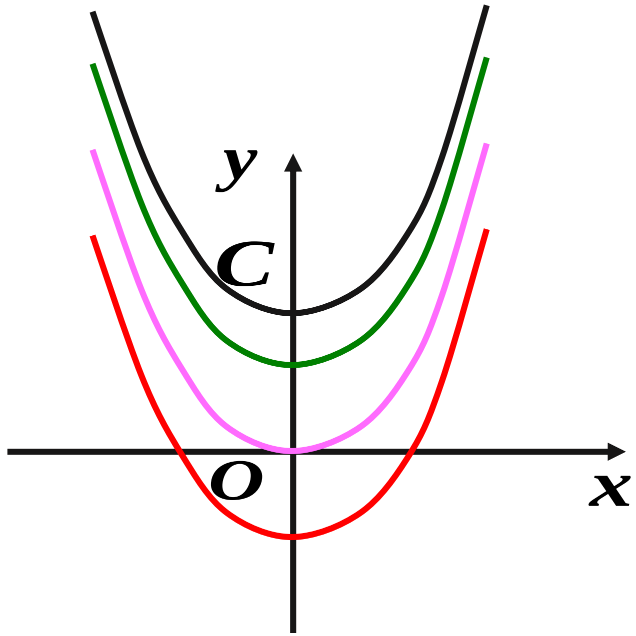
If the curve equation is $y = f(x)$, $\boldsymbol{y' = 2x}$,

An infinite number, eg:

$y = x^2$, $y = x^2 + 1$, $y = x^2 - 1$, $\cdots$

So The desired curve equation is

$\boldsymbol{y = x^2 + C}$, C is any constant.



Definition of Antiderivatives

Antiderivatives :

If on the interval I, $F'(x) = f(x)$ or $\mathrm{d}F(x) = f(x)\mathrm{d}x$,

We call $F(x)$ an antiderivative of $f(x)$ on the interval I

Eg : $(\sin x)' = \cos x$ or $\mathrm{d}\sin x = \cos x\mathrm{d}x$

$F(x) = \sin x$ is an **antiderivative** of $f(x) = \cos x$ on $(-\infty, +\infty)$

C C

$F(x) + C = \sin x + C$ is an antiderivative of $f(x) = \cos x$

where C is any constant.

Theorem

Theorem :

If $F(x)$ is an antiderivative of $f(x)$ on the interval I

Then any antiderivative of $f(x)$ on the interval I can be expressed as

$\boldsymbol{F(x)+C}$, C is any constant.

If $G(x)$ is another antiderivative of $f(x)$, then $G'(x)=f(x)$,

We need to proof : $G(x)-F(x)\equiv$ constant

$F'(x)=f(x)$, and $[G(x)-F(x)]'=G'(x)-F'(x)=f(x)-f(x)\equiv 0$

The function whose derivative is always zero must be a constant

So $G(x)-F(x)=C_0 \Rightarrow G(x)=F(x)+C_0$ any constant

Definition of Indefinite Integral

Indefinite Integral :

If $F(x)$ is any antiderivative of $f(x)$,

Then the general expression of all the antiderivatives of $f(x)$ is $\boldsymbol{F(x)+C}$

It is called the indefinite integral of the function $f(x)$. Recorded as $\int \boldsymbol{f(x)\mathrm{d}x}$

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

Integral sign

Integrand

Integral variables

Integral constant

Example 1

Find $\int x^5 \mathrm{d}x.$

$$\left(\frac{x^6}{6}\right)' = x^5$$

$F(x)$

$$\therefore \int x^5 \mathrm{d}x = \frac{x^6}{6} + C$$

Example 2

Find $\int \frac{1}{1+x^2}\,\mathrm{d}x.$

$F(x)$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$\therefore \quad \int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C$$

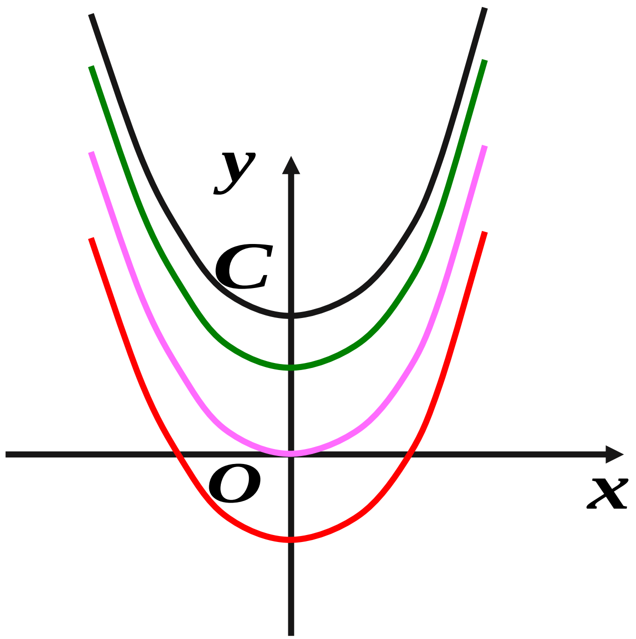
Indefinite Integral

Geometric meaning :

The graph of $y = F(x)$ is a curve of the x, y plane,

and it is called the integral curve of $f(x)$

The graph of $y = F(x) + C$ is the integral curve family of $f(x)$

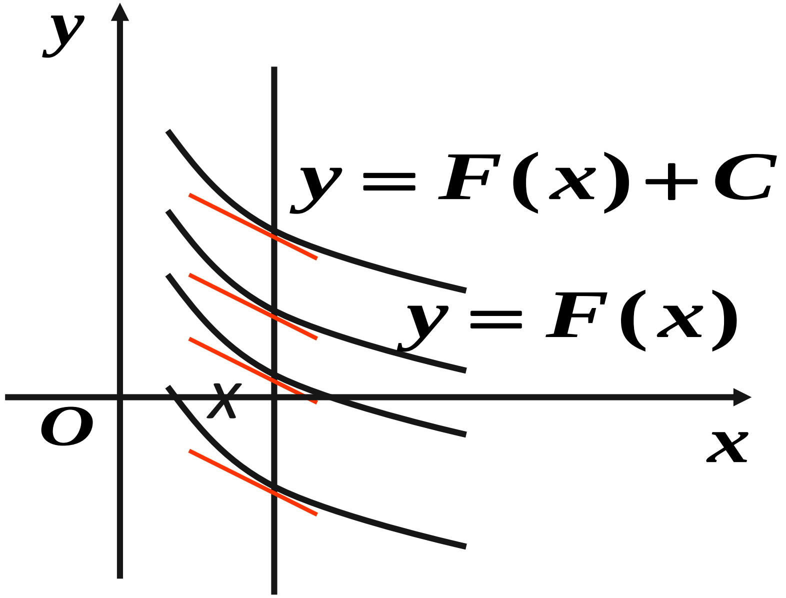


Indefinite Integral

$$[F(x)+C]' = f(x)$$

The derivatives at the same point x are equal to $f(x)$.

Namely, the tangents are parallel to each other.



Example 3

Determination of integral constant

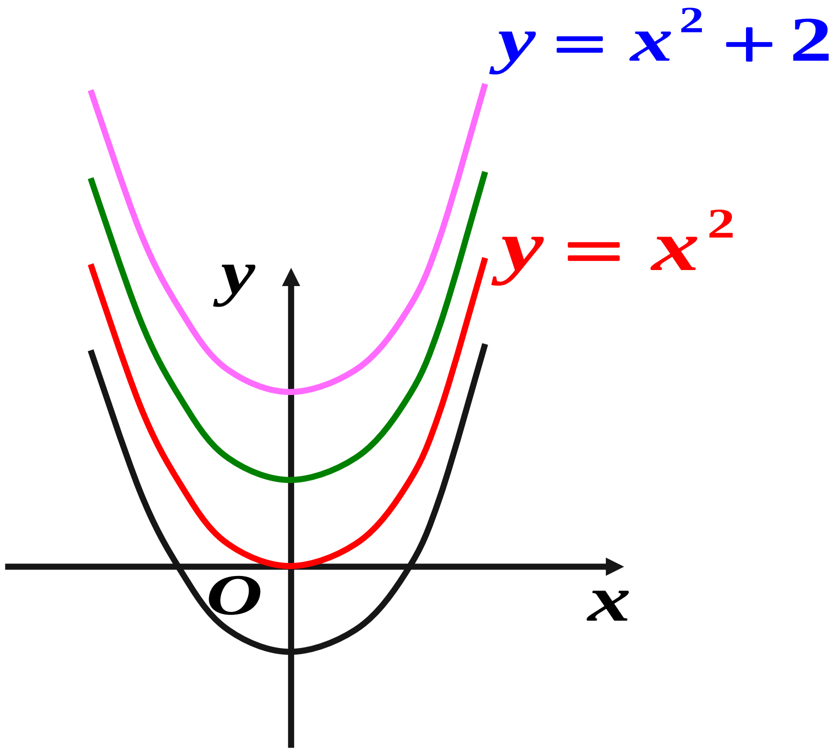
Find the curve that passes through point (2, 6) and whose tangent slope is 2x.

The curve family of $y' = 2x$

is $y = \int 2x\,dx = x^2 + C$

So, $6 = 2^2 + C \Rightarrow C = 2$

Therefore, the desired curve equation is $y = x^2 + 2$



Properties

(1) $\left[\int f(x)\mathrm{d}x\right]_x' = f(x)$ or $\mathrm{d}\left[\int f(x)\mathrm{d}x\right] = f(x)\mathrm{d}x$

$\int F'(x)\,\mathrm{d}x = F(x)+C$ or $\int \mathrm{d}F(x) = F(x)+C$

Conclusion

Differentiation and indefinite integral calculation are reciprocal.

Eg : $\int \mathrm{d}\sin x = \sin x + C,$

(2) $\int [f(x)\pm g(x)]\mathrm{d}x = \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x$

(Extend to a limited number of functions)

(3) $\int kf(x)\mathrm{d}x = k\int f(x)\mathrm{d}x$ (k is a constant, $k \neq 0$)

Note : (2) and (3) are called linear properties.

Theorem

Instance $\left(\frac{x^{\mu+1}}{\mu+1}\right)' = x^{\mu} \Rightarrow \int x^{\mu}dx = \frac{x^{\mu+1}}{\mu+1} + C$

$$(\mu \neq -1)$$

Inspiration Is it possible to derive the integral formula based on the derivation formula ?

Conclusion Integral calculation and differential calculation are reciprocal,

Derivation formula → Integral formula

The Basic Integral Formula

Memorize

(1) $\int k\mathrm{d}x = kx + C$ (k is a constant)

(2) $\int x^{\mu}\mathrm{d}x = \frac{x^{\mu+1}}{\mu+1} + C \quad (\mu \neq -1)$

(3) $\int \frac{\mathrm{d}x}{x} = \ln|x| + C$

(4) $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$

(5) $\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \arcsin x + C$

(6) $\int \cos x\mathrm{d}x = \sin x + C$

(7) $\int \sin x\mathrm{d}x = -\cos x + C$

The Basic Integral Formula

Memorize

$$(8)\quad \int \frac{dx}{\cos^2 x} = \int \sec^2 x\,dx = \tan x + C$$

$$(9)\quad \int \frac{dx}{\sin^2 x} = \int \csc^2 x\,dx = -\cot x + C$$

$$(10)\quad \int \sec x \tan x\,dx = \sec x + C$$

$$(11)\quad \int \csc x \cot x\,dx = -\csc x + C$$

$$(12)\quad \int e^x dx = e^x + C$$

$$(13)\quad \int a^x dx = \frac{a^x}{\ln a} + C$$

Example 4

Direct method

Find the integral $\int x^2\sqrt{x}\,dx$.

$$\int x^2\sqrt{x}\,dx = \int x^{\frac{5}{2}}\,dx$$

By formula $\int x^\mu dx = \frac{x^{\mu+1}}{\mu+1} + C$

$$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + C = \frac{2}{7}x^{\frac{7}{2}} + C$$

Example 5

Find the integral $\int \left(\frac{3}{1+x^2} - \frac{2}{\sqrt{1-x^2}}\right)\mathrm{d}x$.

$$\int \left(\frac{3}{1+x^2} - \frac{2}{\sqrt{1-x^2}}\right)\mathrm{d}x$$

$$= 3\int \frac{1}{1+x^2}\mathrm{d}x - 2\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$$

$$= 3\arctan x - 2\arcsin x + C$$

Example 6

Find the integral $\int \frac{1+x+x^2}{x(1+x^2)}\mathrm{d}x.$

$$\int \frac{1+x+x^2}{x(1+x^2)}\mathrm{d}x = \int \frac{x+(1+x^2)}{x(1+x^2)}\mathrm{d}x$$

$$= \int \left(\frac{1}{1+x^2} + \frac{1}{x}\right)\mathrm{d}x = \int \frac{1}{1+x^2}\mathrm{d}x + \int \frac{1}{x}\mathrm{d}x$$

$$= \arctan x + \ln|x| + C$$

Example 7

Find the integral $\int \frac{1+2x^2}{x^2(1+x^2)}\mathrm{d}x.$

$$\int \frac{1+2x^2}{x^2(1+x^2)}\mathrm{d}x = \int \frac{1+x^2+x^2}{x^2(1+x^2)}\mathrm{d}x$$

Integral by term

$$= \int \frac{1}{x^2}\mathrm{d}x + \int \frac{1}{1+x^2}\mathrm{d}x = -\frac{1}{x} + \arctan x + C$$

Note:

In the last two cases, the integrand is identically transformed, and the linear property is used to calculate the integral, which is called Integral by term.

Example 8

Find the integral $\int \frac{1}{1+\cos 2x} \mathrm{d}x$.

$$\int \frac{1}{1+\cos 2x} \mathrm{d}x = \int \frac{1}{1+2\cos^2 x - 1} \mathrm{d}x$$

$$= \frac{1}{2}\int \frac{1}{\cos^2 x} \mathrm{d}x = \frac{1}{2}\tan x + C$$

Note:

In the above cases, the integrand requires identical deformation in order to use the basic integral table.

Example 9

Find the integral $\int \frac{1}{\sin^2 x \cos^2 x} dx$

$$\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx$$

$$= \tan x - \cot x + C$$

Summary

The concept of the antiderivatives

$$F'(x) = f(x)$$

The concept of the indefinite integral

$$\int f(x)\mathrm{d}x = F(x) + C$$

Geometric meanings, Properties

Reciprocal relationship between differential and integral

Memorize the basic integral formula

Exercise

$$\int \frac{1-x^2}{1+x^2}dx = \int \frac{2-x^2-1}{1+x^2}dx$$

$$\int \tan^2 x dx = \int (\sec^2 x - 1)dx$$

$$\int \sin^2 \frac{x}{2} dx = \int \frac{1-\cos x}{2}dx$$


Antiderivatives